Degree-1 Physics (Hons) Lecture 3 on the topic Damped Harmonic Oscillator.

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Lecture: 3

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Damped Harmonic Oscillator :->

Start with an ideal harmonic oscillator, in which there is no resistance at all,

$$F = -kx$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

where k is force constant

Now
Let's add some resistance we will make the asumption

(i) The force is always opposite to the direction of motion.

(ii) Force (resistive force) depends linearly on the magnitude of the velocity.

$$F = -bv \qquad \text{where b is damping constant}$$

then the sum of forces on the object becomes,

$$m\frac{d^2x}{dt^2} = -kx - bv$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad -(1) \qquad \left(\because v = \frac{dx}{dt}\right)$$

Equation (1) is known as damping equation
Now, the solution of equation is given as.

$$x(t) = A\cos(\omega t + \phi)\, e^{-\frac{bt}{2m}}$$

$$\frac{dx}{dt} = A\left(\frac{-b}{2m}\right)\cos(\omega t + \phi)\, e^{-\frac{bt}{2m}} - A\omega \sin(\omega t + \phi)\, e^{-\frac{bt}{2m}}$$

$$\frac{d^2x}{dt^2} = A\left(\frac{b}{2m}\right)^2 \cos(\omega t + \phi)\, e^{-\frac{bt}{2m}} + \frac{A\omega b}{2m}\sin(\omega t + \phi)\, e^{-\frac{bt}{2m}}$$

$$+ \frac{A\omega b}{2m}\sin(\omega t + \phi)\, e^{-\frac{bt}{2m}} - A\omega^2 \cos(\omega t + \phi)\, e^{-\frac{bt}{2m}}$$

$$\frac{d^2x}{dt^2} = \frac{A\, b^2}{4m^2}\cos(\omega t + \phi)\, e^{-\frac{bt}{2m}} + \frac{2A\omega b}{2m}\sin(\omega t + \phi)\, e^{-\frac{bt}{2m}}$$

$$- A\omega^2 \cos(\omega t + \phi)\, e^{-\frac{bt}{2m}}$$

put the value of $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$ in equation (1)

$$\frac{b^2}{4m} A\cos(\omega t+\phi)\, e^{-\frac{bt}{2m}} + \cancel{A\omega b \sin(\omega t+\phi)\, e^{-\frac{bt}{2m}}} - A\omega^2 m\cos(\omega t+\phi)\, e^{-\frac{bt}{2m}}$$

$$-\frac{b^2}{2m} A\cos(\omega t+\phi)\, e^{-\frac{bt}{2m}} - \cancel{A\omega b \sin(\omega t+\phi)\, e^{-\frac{bt}{2m}}} + KA\cos(\omega t+\phi)\, e^{-\frac{bt}{2m}} = 0$$

$$\left(\frac{b^2}{4m} - \frac{b^2}{2m} - \omega^2 m + K\right) A\cos(\omega t+\phi)\, e^{-\frac{bt}{2m}} = 0$$

$$\because \quad A\cos(\omega t+\phi)\, e^{-\frac{bt}{2m}} \neq 0$$

$$\therefore \quad \frac{b^2}{4m} - \frac{b^2}{2m} - \omega^2 m + K = 0$$

$$-\frac{b^2}{4m} + K = \omega^2 m$$

but $K = m\omega_0^2$

$$\omega^2 m = -\frac{b^2}{4m} + m\omega_0^2$$

where ω_0 is natural frequency

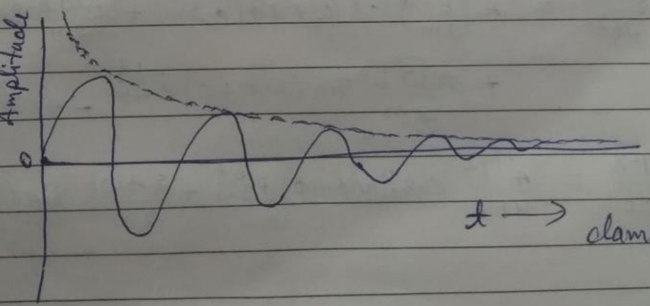
$$\omega^2 = -\frac{b^2}{4m^2} + \omega_0^2$$

$$\boxed{\omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}}$$

ω is the frequency after adding the resistive force.

The function $x(t)$ is not strictly periodic because of the factor $e^{-\frac{bt}{2m}}$ which decrease continuously with time

$$x(t) = A\cos(\omega t+\phi)\, e^{-bt/2m}$$



damped Oscillations

Critical damping ⇒

$$\omega^2 = \omega_0^2 - \frac{b^2}{4m^2}$$

for critical damping $\omega = 0$

$$\omega_0^2 - \frac{b^2}{4m^2} = 0$$

$$\omega_0 = \pm\sqrt{\frac{b^2}{4m^2}}$$

$$\boxed{\omega_0 = \pm \frac{b}{2m}}$$

under this condition the oscillator never really oscillates, ~~of~~ it decay exponentially.