Dr. Mohammad Aslam, Dept. Of physics, B.Sc part-2 physics (Hons) paper-iii lecture no-69

Topic: Propagation of Electromagnetic waves in Anisotropic media

Maxwell's equations in an isotropic media

For the case of a dielectric material, in the absence of free charges, the ME take the form,

$$\nabla \bullet (\varepsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}) = 0 \qquad \nabla . \vec{\mathbf{B}} = 0$$

$$\nabla \text{x} \vec{\mathbf{E}} + \frac{\partial}{\partial t} \vec{\mathbf{B}} = \mathbf{0} \qquad \varepsilon_0 c^2 \nabla \text{x} \vec{\mathbf{B}} = \frac{\partial}{\partial t} (\varepsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}) \tag{1}$$

Here $\vec{\mathbf{P}}$ is the polarizability vector, which accounts for the response of the bound charges to the presence of the electromagnetic wave.

Equation (1) is also typically expressed in terms of the **electric displacement vector** $\vec{\mathbf{D}}$

$$\vec{\mathbf{D}} \equiv \varepsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}} \tag{2}$$

$$\nabla \bullet \vec{\mathbf{D}} = 0 \qquad \nabla . \vec{\mathbf{B}} = 0$$

$$\nabla \text{x} \vec{\mathbf{E}} + \frac{\partial}{\partial t} \vec{\mathbf{B}} = \mathbf{0} \qquad \varepsilon_0 c^2 \nabla \text{x} \vec{\mathbf{B}} = \frac{\partial}{\partial t} \vec{\mathbf{D}} \tag{3}$$

Solution of the Maxwell's equations for the case of anisotropic media

In a non-isotropic material the expression $\mathbf{D} = \varepsilon\,\mathbf{E}$ no longer holds, since $\vec{\mathbf{D}} \equiv \varepsilon_0\vec{\mathbf{E}} + \vec{\mathbf{P}}$ and $\vec{\mathbf{P}}$, in general, is not parallel to $\vec{\mathbf{E}}$. We will still attempt pursuing plane waves as solutions of the ME given in (3).

$$\vec{\mathbf{E}}(\vec{r},t) = \vec{\mathbf{E}}_0 e^{i(\vec{k}\cdot\vec{r}\,-\omega t)}$$
$$\vec{\mathbf{D}}(\vec{r},t) = \vec{\mathbf{D}}_0 e^{i(\vec{k}\cdot\vec{r}\,-\omega t)} \qquad (4)$$
$$\vec{\mathbf{B}}(\vec{r},t) = \vec{\mathbf{B}}_0 e^{i(\vec{k}\cdot\vec{r}\,-\omega t)}$$

where $\vec{k} = k\,\hat{k}$

$\hat{k}$ is a unit vector (whose direction is to be chosen arbitrarily)

k is the magnitude of $\vec{k}$ (so far unknown); $k = |\,\vec{k}\,|$.

The value of k may depend on the particular chosen orientation $\hat{k}$.

We anticipate that in an anisotropic material,

the wave propagation speed v_{wave} *of the plane waves may be different than the energy propagation speed* v_{energy}. (5)

For the moment let's concentrate on finding v_{wave}, which is better expressed as $v_{\text{wave},\,\hat{k}}$, the propagation velocity of the plane waves in the $\hat{k}$ direction.

(In contrast, the energy propagation speed v_{energy} is associated with the direction of the Poynting vector, and will be dealt with later).

The wave velocity $v_{\text{wave},\,\hat{k}}$

Expression (4) describes plane waves propagating with speed $v_{\text{wave},\,\hat{k}} = \dfrac{\omega}{|\vec{k}|}$. The associated index of refraction would be given by,

$$n_{\text{wave},\,\hat{k}} = \frac{c}{v_{\text{wave},\,\hat{k}}}\,.$$

Hence, $n_{\text{wave},\,\hat{k}} = \dfrac{c}{v_{\text{wave},\,\hat{k}}} = \dfrac{c}{\omega}|\,\vec{k}\,|$, or,

$$k = \frac{\omega}{c}\, n_{\text{wave},\,\hat{k}} \qquad (6)$$

In short, the problem we have ahead is the following:

For a given orientation $\hat{k}$, we have to figure out the magnitude of the wave-vector $\vec{k}$, o r equivalently the magnitude of $n_{\text{wave},\,\hat{k}}$, that makes the plane-waves in (4) to become solutions of the Maxwell Equations (3). (7)

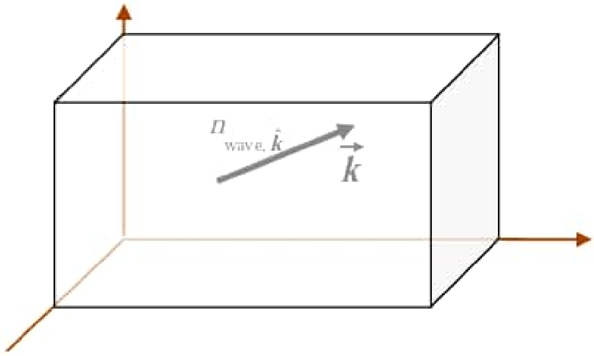


Fig. 1 Propagation of an EM plane wave, with wave-vector $\vec{k}$, in an anisotropic material. We anticipate the magnitude of the wave-vector $|\vec{k}|$ (hence, according to (6), the speed of the plane-wave) to depend on the specific orientation of the unit vector $\hat{k}$.

Orientation of the EM fields $\vec{\mathbf{D}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{E}}$ relative to the wavevector $\vec{k}$

Replacing (4) in (3), the first ME $\nabla \bullet \vec{\mathbf{D}} = 0$ implies,

$$\vec{k} \bullet \vec{\mathbf{D}} = 0 \qquad (8)$$

The second ME $\nabla \bullet \vec{\mathbf{B}} = 0$ implies,

$$\vec{k} \bullet \vec{\mathbf{B}} = 0 \qquad (9)$$

Expressions (8) and (9) indicate that $\vec{\mathbf{D}}$ and $\vec{\mathbf{B}}$ are perpendicular to the wave-vector $\vec{k}$.

However, the latter does not necessarily apply to $\vec{\mathbf{E}}$. That is,

$\vec{k} \bullet \vec{\mathbf{E}} = ?$ (so far unknown)

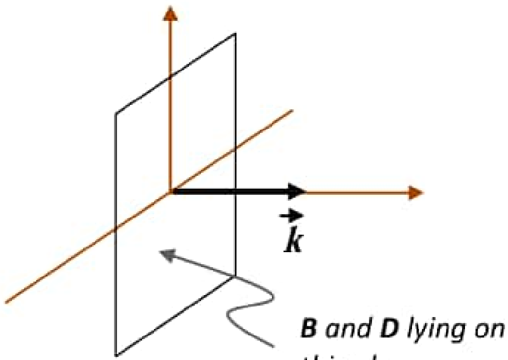


Fig. 2 The particular dependence of the plane wave (4) on the vector $\vec{k}$ will determine (via the ME) to a particular orientation of the EM fields.

The fourth ME $\varepsilon_0 c^2 \nabla \mathrm{x} \vec{\mathbf{B}} = \frac{\partial}{\partial t}\vec{\mathbf{D}}$ implies,

$$\varepsilon_0 c^2 \vec{k} \times \vec{\mathbf{B}} = -\omega \vec{\mathbf{D}} \tag{10}$$

This expression indicates that $\vec{\mathbf{D}}$ is perpendicular to $\vec{k}$ and $\vec{\mathbf{B}}$. One can conveniently choose the orientation of the axis such that,

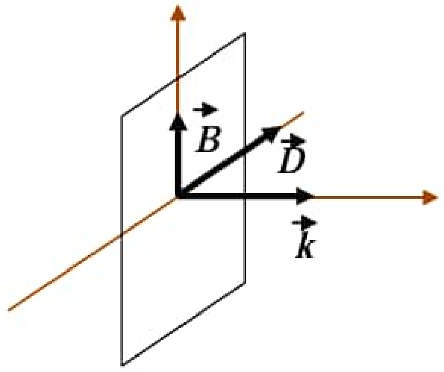
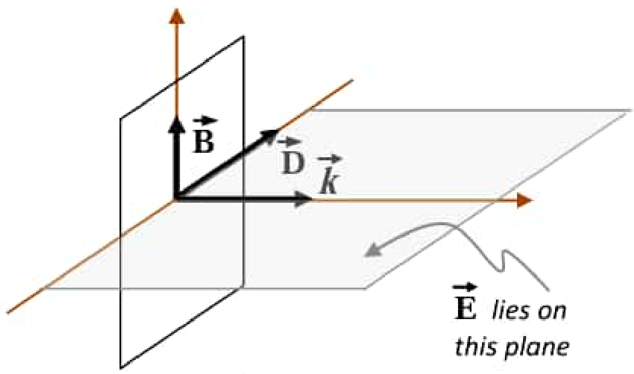


Fig. 3 The particular dependence of the plane wave (4) on the vector $\vec{k}$ has led (via the ME) to this particular orientation of the EM fields.

The third ME $\nabla \mathrm{x} \vec{\mathbf{E}} + \frac{\partial}{\partial t}\vec{\mathbf{B}} = \mathbf{0}$ implies $i\vec{k} \times \vec{\mathbf{E}} - i\omega \vec{\mathbf{B}} = \mathbf{0}$, or

$$\vec{\mathbf{B}} = \frac{1}{\omega}\vec{k} \times \vec{\mathbf{E}} \tag{11}$$

This expression, together with the previous graph, implies that $\vec{\mathbf{E}}$ should lie on the plane defined by $\vec{k}$ and $\vec{\mathbf{D}}$.



$\vec{k} \bullet \vec{\mathbf{D}} = 0 \qquad \vec{k} \bullet \vec{\mathbf{B}} = 0$

$\varepsilon_0 c^2 \vec{k} \times \vec{\mathbf{B}} = -\omega \vec{\mathbf{D}}$

Fig. 4 Orientation of the EM plane-wave fields given in (4) .

$\vec{k}$, $\vec{\mathbf{D}}$ and $\vec{\mathbf{E}}$ are in the same plane (12)

Finding the wave velocity $\text{v}_{\text{wave},\,\hat{k}}$

Now we need to find a relationship between the orientations of the propagation vector $\vec{k}$ of the plane waves given in (4) and the corresponding effective index of refraction $n_{\text{wave},\,\hat{k}}$. We have to find also the orientation of the electric field $\vec{\mathbf{E}}$.

Replacing (11) in (10),

$$\varepsilon_0 c^2 \vec{k}\times(\frac{1}{\omega}\vec{k}\times\vec{\mathbf{E}}) = -\omega\,\vec{\mathbf{D}}$$

$$\vec{k}\times(\vec{k}\times\vec{\mathbf{E}}) = -\frac{\omega^2}{\varepsilon_0 c^2}\vec{\mathbf{D}}$$

$$(\vec{k}\bullet\vec{\mathbf{E}})\,\vec{k} - (\vec{k}\bullet\vec{k})\,\vec{\mathbf{E}} = -\frac{\omega^2}{\varepsilon_0 c^2}\vec{\mathbf{D}}$$

$$\frac{\vec{k}\bullet\vec{\mathbf{E}}}{\vec{k}\bullet\vec{k}}\,\vec{k} - \vec{\mathbf{E}} = -\frac{1}{\vec{k}\bullet\vec{k}}\frac{\omega^2}{\varepsilon_0 c^2}\vec{\mathbf{D}} \tag{13}$$

Notice, for a plane-wave $\vec{\mathbf{E}}(\vec{r},t)=\vec{\mathbf{E}}_0 e^{i(\vec{k}\bullet\vec{r}-\omega t)}$ we expect it to travel with a phase velocity $\text{v}_{\text{wave},\,\hat{k}}=\frac{\omega}{k}$ (where $k^2=\vec{k}\bullet\vec{k}$), to which we associate an index of refraction $n_{\text{wave},\,\hat{k}}=\frac{c}{\text{v}_{\text{wave}}}=\frac{c}{\omega}k$. The notation $n_{\text{wave},\,\hat{k}}$ anticipates that the index of refraction may depend on the orientation of the wave propagation direction.

We summarize the notation we will be using from now on,

$$k=\frac{\omega}{c}n_{\text{wave},\,\hat{k}} \tag{14}$$

$$\text{v}_{\text{wave},\,\hat{k}} = \frac{c}{n_{\text{wave},\hat{k}}}$$

$$\vec{k} = k\,\hat{k}$$

Expression (13) takes the form,

$$(\hat{k}\bullet\vec{\mathbf{E}})\,\hat{k} - \vec{\mathbf{E}} = -\frac{1}{k^2}\frac{\omega^2}{\varepsilon_0 c^2}\vec{\mathbf{D}}$$

$$(\hat{k} \cdot \vec{\mathbf{E}})\,\hat{k} - \vec{\mathbf{E}} = -\frac{1}{\varepsilon_0}\frac{1}{n^2_{\text{wave},\hat{k}}}\vec{\mathbf{D}}$$

Relationship to be fulfilled by the plane waves given in (4) (15)

For each, arbitrarily given, unit vector $\hat{k}$, the value of $n_{\text{wave},\hat{k}}$, as well as $\vec{\mathbf{D}}$ and $\vec{\mathbf{E}}$, need to be determined.

Notice in (15):

- Unless $\vec{\mathbf{E}}$ is perpendicular to $\hat{k}$, the fields $\vec{\mathbf{D}}$ and $\vec{\mathbf{E}}$ are not parallel.
- This equation has been reached by starting from the ME and applied to the case of the plane waves (4).
- This equation is independent of the relationship between $\vec{\mathbf{D}}$ and $\vec{\mathbf{E}}$ demanded by the properties of the medium.