B.Sc. Part-2 Physics (Hons) Lecture-6 on the topic Scalar and Vector Potentials.

B.Sc. Part-2, Physics (Hons)

Lecture – 6

1

Scalar and Vector Potential :→

For static case.

$\vec{\nabla}\times\vec{E} = 0$

we can write $\vec{E} = -\vec{\nabla}V$, where V is called Scalar potential

$\vec{\nabla}\cdot\vec{B} = 0$

Then $\vec{B} = \vec{\nabla}\times\vec{A}$ —— (I)

where $\vec{A}$ is called vector Potential

For Dynamic case

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

Faraday's law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{A})$$

$$\nabla\times\vec{E} + \vec{\nabla}\times\left(+\frac{\partial\vec{B}}{\partial t}\right) = 0$$

$$\nabla\times\left(\vec{E}+\frac{\partial\vec{B}}{\partial t}\right) = 0$$

$$\nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$$

⇓

This can be written as

$$\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V$$

$$\boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}} \quad \text{—— (II)}$$

Gauge Transformations :→

Maxwell's Equation

$$\vec{\nabla}\cdot\vec{E} = +\rho/\epsilon_0 \quad \text{— (a)}$$

but $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$ where ρ is the charge density.

put the value of $\vec{E}$ in equation (a) we get

$$\vec{\nabla}\cdot\left(-\vec{\nabla}V - \frac{\partial A}{\partial t}\right) = \rho/\epsilon_0$$

$$-\nabla^2 V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\epsilon_0$$

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0} \quad \text{— (b)}$$

Now,

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$ (Modified Ampere's law)

where J is the current density

putting $\vec{B} = \vec{\nabla}\times\vec{A}$ and $E = -\vec{\nabla}V - \frac{\partial A}{\partial t}$ in above equation,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial A}{\partial t}\right)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} - \mu_0\epsilon_0\nabla\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$-\nabla^2\vec{A} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J} - \mu_0\epsilon_0\nabla\frac{\partial V}{\partial t} - \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{A})$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \vec{\nabla}\cdot\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right)$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2 A}{\partial t^2} = -\mu_0\vec{J} + \vec{\nabla}\cdot\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right)$$

Here $\frac{1}{c^2} = \mu_0\epsilon_0$

$$\therefore \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \vec{\nabla}\cdot\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) \quad \text{---(c)}$$

The differential Equations (b) and (c) donot uniquely determine the potential V and $\vec{A}$. As perhapes first noted by Lorentz. The magnetic field is unchanged if one adds a gradient of an arbitrary scalar field ϕ.

$$\vec{A} \rightarrow \vec{A}_0 + \vec{\nabla}\phi$$

if we add a time derivative of an arbitrary scalar field ϕ to the scalar potential V_0

$$V \rightarrow V_0 - \frac{\partial \phi}{\partial t}$$

where $\vec{A}_0$ and V_0 are valid electromagnetic potentials, and ϕ is an arbitrary scalar function, now called the gauge tranformation function.

The conditions

$$\left.\begin{aligned} \vec{A} &\rightarrow \vec{A}_0 + \vec{\nabla}\phi \\ V &\rightarrow V_0 - \frac{\partial \phi}{\partial t} \end{aligned}\right\}$$

are called Lorentz Gauge condition.