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 Lecture-10

(1)

Poynting's Theorem \rightarrow

By the Maxwell's Equations.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (a)}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{--- (b)}$$

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \quad \text{--- (c)}$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad \text{--- (d)}$$

Subtracting equation (d) from equation (c)

$$\vec{H} \cdot \vec{\nabla} \times \vec{E} - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \vec{J} - \left(\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

$$\left. \begin{aligned} \therefore \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) &= \vec{\nabla} \cdot (\vec{E} \times \vec{H}) \\ \vec{B} &= \mu \vec{H} \\ \vec{D} &= \epsilon \vec{E} \end{aligned} \right\}$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \vec{J} - \left(\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \vec{J} - \left(\frac{1}{2} \mu \frac{\partial H^2}{\partial t} + \frac{1}{2} \epsilon \frac{\partial E^2}{\partial t} \right)$$

$$\left\{ \frac{\partial H^2}{\partial t} = 2 \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \right\}$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \vec{J} - \frac{\partial}{\partial t} \left(\frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2 \right)$$

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$$\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) = - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{E} \cdot \vec{J}$$

$$U_E = \frac{1}{2} \epsilon E^2 \quad \text{energy density in electric field.}$$

$$U_H = \frac{1}{2} \mu H^2 \quad \text{energy density in magnetic field.}$$

Taking volume integral on both side.

$$\int_V \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) dV = - \int_V \vec{\nabla} \cdot (\vec{E} \times \vec{H}) dV - \int_V \vec{E} \cdot \vec{J} dV$$

Using Gauss' Divergence theorem

$$\int_V \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) dV = - \int_V (\vec{E} \times \vec{H}) \cdot d\vec{A} - \int_V \vec{E} \cdot \vec{J} dV$$

$$\int_V \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) dV = - \int_V \vec{S} \cdot d\vec{A} - \int_V \vec{E} \cdot \vec{J} dV$$

Poynting theorem

$$\text{where } \vec{S} = \vec{E} \times \vec{H}$$

\vec{S} is known as Poynting vector

(Energy Density.)

$$U_E = \frac{1}{2} \epsilon E^2$$

$$U_H = \frac{1}{2} \mu H^2$$

Since the time average of the square of a sine wave.

(3)

$$\vec{E} = E_0 \sin(\omega t + \theta), \quad \vec{H} = H_0 \sin(\omega t + \theta)$$

$$\langle \sin^2 \alpha \rangle = \frac{1}{2}$$

Now the electromagnetic energy density

$$\langle U_E \rangle = \frac{1}{2} \times \frac{1}{2} \epsilon E_0^2$$

$$\langle U_E \rangle = \frac{1}{4} \epsilon E_0^2$$

$$\text{Similarly } \langle U_H \rangle = \frac{1}{4} \mu H_0^2$$

$$\langle U \rangle = \langle U_E \rangle + \langle U_H \rangle$$

$$\langle U \rangle = \left(\frac{1}{4} \epsilon E_0^2 + \frac{1}{4} \mu H_0^2 \right)$$

Now the ratio of the time average energy densities in the electric and magnetic field in the electromagnetic wave is

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \frac{\frac{1}{4} \epsilon E_0^2}{\frac{1}{4} \mu H_0^2}$$

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \frac{\epsilon E_0^2}{\mu \frac{B_0^2}{\mu^2}} \quad \vec{B} = \mu \vec{H}$$

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \mu \epsilon \frac{E_0^2}{B_0^2}$$

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \mu \epsilon v^2$$

$$\text{but } v = \frac{1}{\sqrt{\mu \epsilon}}$$

$$\boxed{\frac{\langle U_E \rangle}{\langle U_H \rangle} = 1}$$

Thus, the energy in an electromagnetic wave is shared equally between the electric and magnetic fields.